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Fisher Information Matrix of Husimi Distribution

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We calculate the Fisher information matrix of Husimi distribution in the Fock–Bargmann representation. It turns out that the Fisher information of the position and that of the momentum move in opposite directions, and that a weighted trace of the Fisher information matrix is a constant independent of the Husimi distribution. This may be interpreted as a kind of uncertainty relation (in the spirit of Heisenberg uncertainty principle) from the statistical inference point of view.

KEY WORDS: Fisher information; Husimi distribution; Fock–Bargmann representation; canonical commutation relation; uncertainty principle.

1. INTRODUCTION

Fisher information, originated in statistical inference[6, 13] for judging the quality of statistical estimates, and defines a Riemann metric on the parameter space,[1, 13] and moreover has a deep connection with probabilistic interpretation of quantum physics.[3] Many thermodynamics can be derived from the standpoint of minimum Fisher information. The traditional measure of disorder, entropy, has provided the usual definitions of time and temperature, now the Fisher information is enjoying growing popularity in a new theory of measurement, since it provides a new dimension to study time and temperature. See Frieden[7] for more details.

In the phase space formulation of quantum mechanics, various distribution functions, such as the Wigner distribution, the Kirkwood distribution, the Glauber–Sudarshan $P-Q$ distribution, and the Husimi distribution, have been introduced. They are related to different ordering of canonical non-commuting pair of operators, and each has its own merits and

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demerits. See refs. 9 or 12 for a review. Among these distributions, the Husimi distribution behaves most regularly and has the simplest and smoothest structure. This can be mostly easily seen from the Fock–Bargmann representation as will be discussed in Section 2.

Carlen[4] discovered a remarkable integral identity for entire functions, and incidentally, found that the trace of the Fisher information of probability density derived from the Bargmann wave function is a constant (independent of the wave function!). Motivated by this, it is tempting to investigate how the Fisher information of the position and the momentum correlate.

In this paper, we first review the fact that the Husimi distribution is precisely the phase space probability derived from the Bargmann wave function. Then we calculate explicitly the Fisher information matrix of any Husimi distribution in the Fock–Bargmann representation, and show how the effect of uncertainty principle manifests in the Fisher information matrix.

To warm up, let us recall the Fisher information of the Schrödinger wave function. Let $L^2(\mathbb{R})$ be the usual Hilbert space in the Schrödinger representation of quantum harmonic oscillator. Let $\psi \in L^2(\mathbb{R})$ be a Schrödinger wave function. According to the postulate of quantum mechanics, $|\psi(x)|^2$ describes the probability density of the position observable, while $|\hat{\psi}(x)|^2$ ($\hat{\psi}$ is the Fourier transform of ψ) describes that of the momentum observable. From the theory of statistical inference, the Fisher information (with respect to the location parameter) of the probability $|\psi(x)|^2$ (or the wave function $\psi(x)$) is

$$\mathbf{I}(\psi) := \int_{\mathbb{R}} (\partial_x |\psi(x)|)^2 \, dx$$

Note that in the usual statistics literature, Fisher information is defined for a parameterized family of probability densities $\{\rho_\theta : \theta \in \mathbb{R}\}$ as

$$\mathbf{I}(\rho_\theta) := \int_{\mathbb{R}} (\partial_\theta \rho_\theta^{1/2}(x))^2 \, dx$$

To be consistent with this general definition, the quantity $\mathbf{I}(\psi)$ should be interpreted as the Fisher information for the parameterized family of probability densities $\{\rho_\theta(x) := |\psi(x+\theta)|^2 : \theta \in \mathbb{R}\}$, that is,

$$\mathbf{I}(\psi) = \mathbf{I}(\rho_\theta)$$

This identity is guaranteed by the facts that $\partial_x |\psi(x)| = \partial_\theta \rho_\theta^{1/2}(x)$ and the translation invariance of Lebesgue integral.

By the principle of statistical inference, Fisher information describes the quantitative information in inferring the parameter, here the position observable. The most famous result concerning Fisher information is the classical Cramér–Rao inequality:[13]

$$\mathrm{Var}_\theta\, \hat{\theta} \geqslant \frac{1}{\mathbf{I}(\rho_\theta)}$$

Here ρ_θ is a parameterized family of probability densities, $\hat{\theta}$ is any unbiased estimator for the parameter θ, and $\mathrm{Var}_\theta\, \hat{\theta}$ denotes the variance (under the probability ρ_θ) of the estimator $\hat{\theta}$.

Cramér–Rao inequality is a kind of uncertainty relation for estimation, resembling the Heisenberg uncertainty principle. Moreover, it actually implies the Heisenberg uncertainty relation.[3, 7]

In particular, for the Gaussian wave packet

$$\psi(x) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-m)^2/(2\sigma^2)} \right)^{1/2}$$

its Fisher information (with respect to the location parameter) is

$$\mathbf{I}(\psi) = \frac{1}{4\sigma^2}$$

Thus $\mathbf{I}(\psi)$ depends on the variance σ^2, and its value range is $[0, \infty)$ when σ varies. In general, $\mathbf{I}(\psi)$ depends on the wave function ψ.

Following Husimi[10] or Lee,[12] for any Schrödinger wave function $\psi \in L^2(\mathbb{R})$, its Husimi distribution is defined as

$$H_\psi(q, p) = |\langle \psi, h_{s;\, q,\, p} \rangle|^2$$

which is a probability density on the phase space $(\mathbb{R} \times \mathbb{R}, (2\pi\hbar)^{-1}\, dq\, dp)$. Here $s > 0$ is an arbitrary parameter, and

$$h_{s;\, q,\, p}(x) := (2\pi s^2)^{-1/4}\, e^{-(x-q)^2/(4s^2)} e^{ipx/\hbar}$$

In the limit $s \to 0$, the minimal uncertainty function $h_{s;\, q,\, p}$ becomes concentrated in position, thus approximates a position eigenfunction. Alternatively, when $s \to \infty$, it approximates a momentum eigenfunction. In this paper, we take the convention that the Hilbert space inner product to be complex linear in the first variable.

The Schrödinger wave function only describes the position observable or the momentum observable, but not both simultaneously. On the other hand, the Husimi distribution is defined on the phase space, thus describes in certain sense the "joint probability" of the position observable and the momentum observable. Of course, Heisenberg uncertainty principle makes it impossible to take this interpretation literally.

2. HUSIMI DISTRIBUTION IN FOCK–BARGMANN REPRESENTATION

In the Fock–Bargmann representation of quantum harmonic oscillator with one degree of freedom,[2] the state space is

$$H^2(\mathbb{C}) := \left\{ f : \mathbb{C} \to \mathbb{C}, \text{ holomorphic}, \langle f, f \rangle := \int_{\mathbb{C}} f(z)\, \overline{f(z)}\, d\mu(z) < \infty \right\}$$

Here $d\mu(z) = \pi^{-1} e^{-z\bar{z}}\, dz\, d\bar{z}$ is the standard Gaussian measure on the phase space $\mathbb{C}$. $H^2(\mathbb{C})$ has reproducing kernel $e_w(z) := e^{\bar{w}z}$, that is, for any $f \in H^2(\mathbb{C})$, we have

$$f(z) = \int_{\mathbb{C}} e^{\bar{w}z} f(w)\, d\mu(w), \qquad \text{for any} \quad z \in \mathbb{C}$$

The set of coherent states $\{e_w : w \in \mathbb{C}\}$ is total in $H^2(\mathbb{C})$.

The annihilation operator a^- and the creation operator a^+ are defined as

$$a^- f(z) := \frac{\partial}{\partial z} f(z), \qquad a^+ f(z) := z(f)$$

respectively. The operators a^- and a^+ are adjoint to each other on $H^2(\mathbb{C})$, and satisfy the canonical commutation relation:

$$[a^-, a^+] := a^- a^+ - a^+ a^- = I$$

The Schrödinger representation and the Fock–Bargmann representation are unitarily equivalent. They are related by the Bargmann transform

$$B_s \psi(z) = \int_{\mathbb{R}} \varepsilon_{s,z}(x)\, \psi(x)\, dx$$

which establishes an isometry from $L^2(\mathbb{R})$ onto $H^2(\mathbb{C})$. Here

$$\varepsilon_{s,z}(x) := (2\pi s^2)^{-1/4} \exp\{-z^2/2 + zx/s - x^2/(4s^2)\}$$

is the coherent state in $L^2(\mathbb{R})$ parameterized by $z \in \mathbb{C}$, the phase space, and $s > 0$ is an arbitrary parameter. Note that the set of coherent states $\{\varepsilon_{s,w} : w \in \mathbb{C}\}$ is total in $L^2(\mathbb{R})$, and $B_s \varepsilon_{s,w}(z) = e^{wz}$.

For any $\psi \in L^2(\mathbb{R})$ with unit norm, let $f(z) = B_s\psi(z)$ (note that f depends on the parameter s only through the dependence of z on s, so it is natural to leave the subscript s off f). Then

$$p_f(z) := |f(z)|^2 e^{-|z|^2}$$

is a probability density on $(\mathbb{C}, \pi^{-1} dz\, d\bar{z})$. Put

$$z = \frac{1}{2s} q - i \frac{s}{\hbar} p$$

Then

$$h_{s;\, q,\, p}(x) = \varepsilon_{s,\, \bar{z}}(x)\, e^{-|z|^2/2} e^{ipq/2\hbar}$$

Consequently,

$$H_\psi(q, p) = |\langle \psi, h_{s;\, q,\, p} \rangle|^2 = |\langle \psi, \varepsilon_{s,\, \bar{z}} \rangle|^2 e^{-|z|^2} = p_f(z)$$

Thus, the Husimi distribution of ψ is precisely the phase space probability derived from the Bargmann wave function $f = B_s\psi$ if we parameterize the complex coordinate z as $z = (1/2s)\, q - i(s/\hbar)\, p$.

3. FISHER INFORMATION MATRIX OF HUSIMI DISTRIBUTION

For $f \in H^2(\mathbb{C})$, define a nonlinear operator A as

$$Af(z) = \frac{(a^- f(z))^2}{f(z)}$$

and two quadratic Hamiltonians Q and P as

$$Q = \frac{(a^-)^2 + (a^+)^2}{2}, \qquad P = \frac{(a^-)^2 - (a^+)^2}{2i}$$

respectively.

For any $\psi \in L^2(\mathbb{R})$ with unit norm, let $f(z) = B_s\psi(z)$. We want to calculate the Fisher information matrix of the Husimi distribution

$$H_\psi(q, p) = p_f(z) = |f(z)|^2 e^{-|z|^2}$$

in the complex coordinate.

The Fisher information matrix of ρ_f is by definition the 2×2 matrix

$$\mathbf{I}(f)=\int_{\mathbb{C}}(\nabla\rho_f^{1/2}(z))^t\,(\nabla\rho_f^{1/2}(z))\,\pi^{-1}\,dz\,d\bar{z}$$

Here $z=(1/2s)\,q-i(s/\hbar)\,p$, $\nabla=(\partial_q,\partial_p)$ is the gradient operator, and t denotes transpose.

Theorem 1. For any $\psi\in L^2(\mathbb{R})$ with unit norm, let $f=B_s\psi\in H^2(\mathbb{C})$ be its Bargmann transform. Put

$$\alpha=\langle Af,f\rangle-\langle(a^-)^2 f,f\rangle$$

Then

$$\mathbf{I}(f)=\frac{1}{2}\begin{pmatrix}\dfrac{1}{4s^2}(1+\mathrm{Re}\,\alpha) & \dfrac{1}{2\hbar}\,\mathrm{Im}\,\alpha\\[2ex] \dfrac{1}{2\hbar}\,\mathrm{Im}\,\alpha & \dfrac{s^2}{\hbar^2}(1-\mathrm{Re}\,\alpha)\end{pmatrix}$$

provided that the integrals exist. Here Re and Im denote the real part and imaginary part of a complex number, respectively.

Proof. Note that $\partial_q f(z)=(1/2s)\,a^-f(z)$ and $\partial_p f(z)=-i(s/\hbar)\,a^-f(z)$ for $z=(1/2s)\,q-i(s/\hbar)\,p$, and

$$\nabla\rho_f^{1/2}(z)=(\partial_q\rho_f^{1/2}(z),\,\partial_p\rho_f^{1/2}(z))$$

we have

$$\begin{aligned}\partial_q\rho_f^{1/2}(z)&=\partial_q((f(z)\,\overline{f(z)})^{1/2}\,e^{-|z|^2/2})\\&=\frac{1}{2}(f(z)\,\overline{f(z)})^{-1/2}\,[\overline{f(z)}\,\partial_q f(z)+f(z)\,\partial_q\overline{f(z)}]\,e^{-|z|^2/2}\\&\quad-(f(z)\,\overline{f(z)})^{1/2}\,\frac{1}{2s}\cdot\frac{z+\bar{z}}{2}\,e^{-|z|^2/2}\\&=\frac{1}{2}(f(z)\,\overline{f(z)})^{-1/2}\,\frac{1}{2s}\,[\overline{f(z)}\,a^-f(z)+f(z)\,\overline{a^-f(z)}]\,e^{-|z|^2/2}\\&\quad-\frac{1}{2}\cdot\frac{1}{2s}(f(z)\,\overline{f(z)})^{1/2}\,(z+\bar{z})\,e^{-|z|^2/2}\end{aligned}$$

and similarly,

$$\partial_p \rho_f^{1/2}(z) = \partial_p((f(z)\,\overline{f(z)})^{1/2}\, e^{-|z|^2/2})$$
$$= \frac{1}{2}(f(z)\,\overline{f(z)})^{-1/2}\,[\,\overline{f(z)}\,\partial_p f(z) + f(z)\,\partial_p \overline{f(z)}\,]\, e^{-|z|^2/2}$$
$$-(f(z)\,\overline{f(z)})^{1/2}\,\frac{s}{\hbar}\cdot\frac{z-\bar{z}}{2i}\, e^{-|z|^2/2}$$
$$= -\frac{1}{2}(f(z)\,\overline{f(z)})^{-1/2}\,\frac{s}{\hbar}\,[\,\overline{f(z)}\, ia^- f(z) - if(z)\,\overline{a^- f(z)}\,]\, e^{-|z|^2/2}$$
$$+\frac{i}{2}(f(z)\,\overline{f(z)})^{1/2}\,\frac{s}{\hbar}\,(z-\bar{z})\, e^{-|z|^2/2}$$

Thus noting that a^- and a^+ are adjoint to each other, by direct calculation, we have

$$\mathbf{I}_{11}(f) := \int_{\mathbb{C}} (\partial_q \rho_f^{1/2})^2\, \pi^{-1}\, dz\, d\bar{z}$$
$$= \frac{1}{2}\cdot\frac{1}{4s^2}\,(\mathrm{Re}\langle Af, f\rangle - \langle Qf, f\rangle + \langle f, f\rangle)$$
$$= \frac{1}{2}\cdot\frac{1}{4s^2}\,(1 + \mathrm{Re}\langle Af, f\rangle - \langle Qf, f\rangle) = \frac{1}{2}\cdot\frac{1}{4s^2}\,(1 + \mathrm{Re}\,\alpha)$$

$$\mathbf{I}_{22}(f) := \int_{\mathbb{C}} (\partial_p \rho_f^{1/2})^2\, \pi^{-1}\, dz\, d\bar{z}$$
$$= \frac{1}{2}\cdot\frac{s^2}{\hbar^2}\,(-\mathrm{Re}\langle Af, f\rangle + \langle Qf, f\rangle + \langle f, f\rangle)$$
$$= \frac{1}{2}\cdot\frac{s^2}{\hbar^2}\,(1 - \mathrm{Re}\langle Af, f\rangle + \langle Qf, f\rangle) = \frac{1}{2}\cdot\frac{s^2}{\hbar^2}\,(1 - \mathrm{Re}\,\alpha)$$

$$\mathbf{I}_{12}(f) := \int_{\mathbb{C}} (\partial_q \rho_f^{1/2})(\partial_p \rho_f^{1/2})\, \pi^{-1}\, dz\, d\bar{z}$$
$$= \frac{1}{2}\cdot\frac{1}{2\hbar}\,(\mathrm{Im}\langle Af, f\rangle - \langle Pf, f\rangle) = \frac{1}{2}\cdot\frac{1}{2\hbar}\,\mathrm{Im}\,\alpha$$

Note that $\mathbf{I}_{12}(f) = \mathbf{I}_{21}(f)$, we obtain the Fisher information matrix.

Corollary 2. Under the assumption of Theorem 1, it holds that

$$4s^2\mathbf{I}_{11}(f) + \frac{\hbar^2}{s^2}\mathbf{I}_{22}(f) = 1$$

The above fact (with $2s = 2\hbar = 1$) was first discovered by Carlen,[4] who actually proved a more general integral identity. However, the way we come it is quite different from that of Carlen.

Let us see some particular cases.

Example 1. For any $w \in \mathbb{C}$, let

$$\psi(x) = e^{-|w|^2/2}\varepsilon_{s,w}(x) = (2\pi)^{-1/4}\exp\{-w^2/2 + wx/s - x^2/(4s^2) - |w|^2/2\}$$

be the normalized coherent state in $L^2(\mathbb{R})$, then $f(z) = B_s\psi(z) = e^{-|w|^2/2 + \bar{w}z}$ is a normalized coherent state in $H^2(\mathbb{C})$. Simple calculation shows that its Fisher information matrix is

$$\mathbf{I}(f) = \frac{1}{2}\begin{pmatrix} \frac{1}{4s^2} & 0 \\ 0 & \frac{s^2}{\hbar^2} \end{pmatrix}$$

Example 2. Let $\psi(x) = (2\pi s^2)^{-1/4}(n!)^{-1/2}e^{-x^2/(4s^2)}(\partial^n/\partial z^n) \times e^{-z^2/2 + zx/s}|_{z=0} \in L^2(\mathbb{R})$ be a normalized Hermite function of order n with a scaling parameter s, then $f(z) = B_s\psi(z) = z^n/\sqrt{n!} \in H^2(\mathbb{C})$ is a normalized monomial. Simple calculation also leads to

$$\mathbf{I}(f) = \frac{1}{2}\begin{pmatrix} \frac{1}{4s^2} & 0 \\ 0 & \frac{s^2}{\hbar^2} \end{pmatrix}$$

It seems that the coherent states and monomials are the only Bargmann wave functions that possess the above Fisher information matrix. But we fail to prove or disprove this speculation. However, when considering this conjecture, it maybe useful to note that by Theorem 1,

$$\mathbf{I}(f) = \frac{1}{2}\begin{pmatrix} \frac{1}{4s^2} & 0 \\ 0 & \frac{s^2}{\hbar^2} \end{pmatrix}$$

is equivalent to $\alpha = 0$, and that α can be rewritten as

$$\alpha = -\langle (a^-)^2 \log f \cdot f, f \rangle$$

Example 3. Let $\psi \in L^2(\mathbb{R})$ have the Bargmann transform $f(z) = B_s\psi(z) = cze^{\bar{\beta}z}$, $\beta \in \mathbb{C}$. Here $c = (1+\beta\bar{\beta})^{-1/2}\, e^{-\beta\bar{\beta}/2}$ is the normalization constant such that f has unit norm. We will calculate the Fisher information matrix of this f.

Note that $f = ca^+ e_\beta$, we have

$$\begin{aligned}\langle (a^+)^2 f, f\rangle &= c^2 \langle (a^+)^3 e_\beta, a^+ e_\beta\rangle \\ &= c^2\langle a^-(a^+)^3 e_\beta, e_\beta\rangle \\ &= c^2\langle ((a^+)^3 a^- + 3(a^+)^2)\, e_\beta, e_\beta\rangle \\ &= c^2(\beta^3\bar{\beta} + 3\beta^2)\, e^{\beta\bar{\beta}}\end{aligned}$$

Since $\langle (a^-)^2 f, f\rangle = \langle f, (a^+)^2 f\rangle = \overline{\langle (a^+)^2 f, f\rangle}$, we obtain

$$\langle Qf, f\rangle = \frac{c^2}{2}(\beta^2 + \bar{\beta}^2)(3+\beta\bar{\beta})\, e^{\beta\bar{\beta}},$$

$$\langle Pf, f\rangle = \frac{c^2}{2i}(\bar{\beta}^2 - \beta^2)(3+\beta\bar{\beta})\, e^{\beta\bar{\beta}}$$

Now from

$$Af(z) = c\,\frac{(e^{\bar{\beta}z} + \bar{\beta}ze^{\bar{\beta}z})^2}{ze^{\bar{\beta}z}} = c\,\frac{(1+\bar{\beta}z)^2}{z}\, e^{\bar{\beta}z}$$

we have

$$\begin{aligned}c^{-2}\langle Af, f\rangle &= \int_{\mathbb{C}} \frac{(1+\bar{\beta}z)^2}{z}\, e^{\bar{\beta}z}\bar{z}e^{\beta\bar{z}}\, d\mu(z) \\ &= \int_{\mathbb{C}} \left(\frac{\bar{z}}{z} + 2\bar{\beta}z + \bar{\beta}^2 z\bar{z}\right) e^{\bar{\beta}z+\beta\bar{z}}\, d\mu(z) \\ &= \int_{\mathbb{C}} \frac{\bar{z}}{z}\, e^{\bar{\beta}z+\beta\bar{z}}\, d\mu(z) + \int_{\mathbb{C}} 2\bar{\beta}ze^{\bar{\beta}z+\beta\bar{z}}\, d\mu(z) + \int_{\mathbb{C}} \bar{\beta}^2 z\bar{z}e^{\bar{\beta}z+\beta\bar{z}}\, d\mu(z) \\ &= J_1 + J_2 + J_3 \quad \text{(say)}\end{aligned}$$

By series expansion and noting orthogonality, we have

$$
\begin{aligned}
J_1 &= \sum_{n=0}^{\infty} \int_{\mathbb{C}} \frac{\bar{z}}{z} \cdot \frac{(\bar{z}+\beta\bar{z})^n}{n!}\, d\mu(z) \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{C}} \frac{\bar{z}}{z} \frac{\binom{2k}{k+1}(\bar{\beta}z)^{k+1}\,(\beta\bar{z})^{k-1}}{(2k)!}\, d\mu(z) \\
&= \sum_{k=1}^{\infty} \frac{\bar{\beta}^{k+1}\beta^{k-1}}{(k+1)(k-1)!} \\
&= \bar{\beta}^2\left(\frac{1}{\beta\bar{\beta}}\, e^{\beta\bar{\beta}} - \frac{1}{(\beta\bar{\beta})^2}\, e^{\beta\bar{\beta}} + \frac{1}{(\beta\bar{\beta})^2}\right) \\
J_2 &= \sum_{n=0}^{\infty} \int_{\mathbb{C}} 2\bar{\beta}\bar{z}\, \frac{(\bar{\beta}z+\beta\bar{z})^n}{n!}\, d\mu(z) \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{C}} 2\bar{\beta}\bar{z}\, \frac{\binom{2k+1}{k+1}(\bar{\beta}z)^{k+1}\,(\beta\bar{z})^k}{(2k+1)!}\, d\mu(z) \\
&= \sum_{k=0}^{\infty} 2\bar{\beta}\, \frac{\bar{\beta}^{k+1}\beta^k}{k!} = 2\bar{\beta}^2 e^{\bar{\beta}\bar{\beta}} \\
J_3 &= \sum_{n=0}^{\infty} \int_{\mathbb{C}} \bar{\beta}^2 z\bar{z}\, \frac{(\bar{\beta}z+\beta\bar{z})^n}{n!}\, d\mu(z) \\
&= \sum_{k=0}^{\infty} \bar{\beta}^2 z\bar{z}\, \frac{\binom{2k}{k}(\bar{\beta}z)^k\,(\beta\bar{z})^k}{(2k)!}\, d\mu(z) \\
&= \sum_{k=0}^{\infty} \frac{(k+1)\,\bar{\beta}^{k+2}\beta^k}{k!} \\
&= \bar{\beta}^2(1+\beta\bar{\beta})\, e^{\beta\bar{\beta}}
\end{aligned}
$$

Consequently,

$$
\langle Af, f\rangle = c^2\bar{\beta}^2\left(\frac{1}{\beta\bar{\beta}}\, e^{\beta\bar{\beta}} - \frac{1}{(\beta\bar{\beta})^2}\, e^{\beta\bar{\beta}} + \frac{1}{(\beta\bar{\beta})^2} + 3e^{\beta\bar{\beta}} + \beta\bar{\beta}e^{\beta\bar{\beta}}\right)
$$

and thus

$$
\begin{aligned}
\operatorname{Re}\langle Af, f\rangle &= \frac{c^2}{2}\,(\bar{\beta}^2+\beta^2)\left(\left(\frac{1}{\beta\bar{\beta}} - \frac{1}{(\beta\bar{\beta})^2} + 3 + \beta\bar{\beta}\right) e^{\beta\bar{\beta}} + \frac{1}{(\beta\bar{\beta})^2}\right) \\
\operatorname{Im}\langle Af, f\rangle &= \frac{c^2}{2i}\,(\bar{\beta}^2-\beta^2)\left(\left(\frac{1}{\beta\bar{\beta}} - \frac{1}{(\beta\bar{\beta})^2} + 3 + \beta\bar{\beta}\right) e^{\beta\bar{\beta}} + \frac{1}{(\beta\bar{\beta})^2}\right)
\end{aligned}
$$

Finally, we obtain the Fisher information matrix with entries

$$\mathbf{I}_{11}=\left(\frac{1}{2}+\frac{c^2}{4}(\bar{\beta}^2+\beta^2)\left(\left(\frac{1}{\beta\bar{\beta}}-\frac{1}{(\beta\bar{\beta})^2}\right)e^{\beta\bar{\beta}}+\frac{1}{(\beta\bar{\beta})^2}\right)\right)\frac{1}{4s^2}$$

$$\mathbf{I}_{22}=\left(\frac{1}{2}-\frac{c^2}{4}(\bar{\beta}^2+\beta^2)\left(\left(\frac{1}{\beta\bar{\beta}}-\frac{1}{(\beta\bar{\beta})^2}\right)e^{\beta\bar{\beta}}+\frac{1}{(\beta\bar{\beta})^2}\right)\right)\frac{s^2}{\hbar^2}$$

$$\mathbf{I}_{12}=\mathbf{I}_{21}=\left(\frac{c^2}{4i}(\beta^2-\bar{\beta}^2)\left(\left(\frac{1}{\beta\bar{\beta}}-\frac{1}{(\beta\bar{\beta})^2}\right)e^{\beta\bar{\beta}}+\frac{1}{(\beta\bar{\beta})^2}\right)\right)\frac{1}{2\hbar}$$

4. DISCUSSION

We have seen, in sharp contrast to the Schrödinger representation, in which the Fisher information of wave function does not have an upper bound, a weighted trace of the Fisher information matrix (which is a linear combination of the Fisher information of the position wave function and the momentum wave function) of Husimi distribution (or equivalently, of probability distribution derived from the Bargmann wave function) is a constant independent of the wave function, thus has an upper bound. Since the Schrödinger wave function only describes the position or the momentum (but not both) probability density, while the Husimi distribution describes in certain sense the “joint probability” of the position and momentum, the result of present article may be viewed as a kind of uncertainty relation.

Physically, loss of information is recognized as the gain of entropy. Actually, the logarithmic Sobolev inequality of Gross[8] relates the Fisher information and the Shannon entropy. Thus the upper bound of Fisher information induces naturally a low bound for the (Shannon) entropy of the probability derived from the Bargmann wave function, and consequently, Fisher information also enters the scenario of maximum entropy principle of Jaynes.[11] Carlen[5] first presented a beautiful proof of Wehrl's conjecture on classical entropy by virtue of this idea, and more generally, derived the logarithmic Sobolev inequality from superadditivity of the Fisher information.

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